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Scale-covariant field theories: III. The augmented scalar theory

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Abstract. We examine the augmented formalism for a scalar pseudo-free field theory. Motivated by a diagrammatic expansion for mass renormalisation we justify a constraint equation for the theory that gives additional information over the subtracted scale-invariant formalism.

1. Introduction

As has become apparent from our previous papers (Ebbutt and Rivers 1982a, b, to be referred to as I and II, respectively), in general we do not know how to solve analytically for scale-covariant field theories (Klauder 1978, 1979). In such a situation we try to get some insight by examining particular (and less intractable) examples. The case that we shall consider here is that of a scalar theory with scale-*invariant* measure.

In I and II we suggested that the augmented *translation-covariant* reformulation (Klauder 1977) of such a theory gives rise to the least degenerate branching equations between Green functions. This was based on formal manipulations of path integrals, a notoriously unreliable practice.

In this paper we shall examine the augmented formalism more carefully. Our initial aim is to show in more detail how the naive augmented theory equations are indeed correct.

It is sufficient for this purpose to work with the *pseudo-free* theory of a single scalar field φ. The generating functional for the Minkowski theory is (Klauder 1977)†

$$Z_0'[h]=\int \mathscr{D}[\varphi]\mathscr{D}[\chi]\exp \mathrm{i}\int \mathrm{d}x\,[\tfrac{1}{2}(\partial_\mu\varphi)^2-\tfrac{1}{2}m_0^2\varphi^2-\tfrac{1}{2}\eta\varphi^2\chi^2+h\varphi] \tag{1.1}$$

where $\mathscr{D}[\varphi]$, $\mathscr{D}[\chi]$ are translationally invariant measures normalised so that $Z_0'[0]=1$.

On integrating over the auxiliary field χ we recover the original expression for the pseudo-free theory (Klauder 1979)

$$Z_0'[h]=\int \mathscr{D}'[\varphi]\exp \mathrm{i}\int \mathrm{d}x\,[\tfrac{1}{2}(\partial_\mu\varphi)^2-\tfrac{1}{2}m_0^2\varphi^2+h\varphi] \tag{1.2}$$

in terms of the scale-invariant measure $\mathscr{D}'[\varphi]$.

† In II, where we were primarily interested in combinatorics we used the Euclidean theory. Here we use the Minkowski theory.

However, the formalism (1.1) permits an alternative description obtained by performing the φ integration first. This gives

$$Z_0'[h]=\int \mathscr{D}[\chi]\exp \mathrm{i}\mathfrak{A}_0[\chi, h] \tag{1.3}$$

where

$$\mathfrak{A}_0[\chi, h]=-\frac{\mathrm{i}}{2}\int \mathrm{d}x\ \mathrm{d}y h(x)g(x, y; \chi)h(y)+\tfrac{1}{2}\mathrm{i}\ \mathrm{Tr}\ \ln(\Box+m_0^2+\eta\chi^2) \tag{1.4}$$

with

$$(\Box_x+m_0^2+\eta\chi^2(x))g(x, y; \chi)=-\mathrm{i}\delta(x-y). \tag{1.5}$$

The advantage of (1.3) over (1.2) is that it lends itself to a diagrammatic expansion, which is the content of the next section. This enables us, in §§ 3 and 4, to establish a relationship between normal-ordering and the subtraction procedure (Klauder 1979) in the scale-covariant formalism that caused us such concern in II. In the process of doing so, we get some understanding of the way mass renormalisation arises in the theory. This is important, since our understanding of renormalisation in scale-covariant theories is very poor.

To provide additional support to this analysis, we use § 5 to re-examine the augmented branching equations for solutions compatible with the above.

In § 6 we repeat part of the analysis for the large-N limit of the $O(N)$ pseudo-free theory.

Our conclusions are presented in the final section.

2. Diagrammatic description of the pseudo-free theory

Given that the translation-covariant augmented formalism of (1.1) is of a nature to permit diagrammatic expansions, what do we expect?

Klauder (1979) has suggested that a scale-invariant measure is somehow equivalent to introducing a partial 'hard-core' interaction into the theory. Thus the pseudo-free theory will give rise to scattering (and will have non-trivial connected Green functions W_{2n} for arbitrarily large n). Moreover, if the consequences of changing measure in quantum mechanics are any guide, the 'hard-core' interaction will be effectively expressed as a non-polynomial interaction† (Kay 1981).

It may be argued that this is an unfortunate way to look at a scale-invariant theory, since describing the hard-core effect of the measure as a non-polynomial interaction is, in effect, to attempt an expansion about the *free* scalar theory. This seems to run counter to the spirit of scale-covariant theories, even though our understanding is so incomplete that we are reluctant to foreclose any avenue of approach at this state.

Despite this unease, we proceed. Introducing a source $j(x)$ coupled to $\chi(x)$ in (1.1) enables us to define the generating functional

$$Z_0'[h, j]=\int \mathscr{D}[\varphi]\mathscr{D}[\chi]\exp \mathrm{i}\int \mathrm{d}x[\tfrac{1}{2}(\partial_\mu\varphi)^2-\tfrac{1}{2}m_0^2\varphi^2-\tfrac{1}{2}\eta\varphi^2\chi^2+h\varphi+j\chi] \tag{2.1}$$

$$=\int \mathscr{D}[\chi]\exp\left(\mathrm{i}\mathfrak{A}_0[\chi; h]+\mathrm{i}\int j\chi\right) \tag{2.2}$$

† This follows from approximating finite step potentials by large order polynomials in Kay (1981).

in terms of $\mathfrak{A}_0$ of (1.4). The generating functional $Z_0'[h]$ of (1.1) is obviously given by $Z_0'[h]=Z_0'[h, j=0]$.

Let $G_0(x-y)$ be the free-field Green function satisfying

$$(\Box_x + m_0^2)G_0(x-y) = -\mathrm{i}\delta(x-y). \tag{2.3}$$

We then separate out the free-field functional

$$Z_0[h] = \exp \mathrm{i} W_0[h] \tag{2.4}$$

with

$$W_0[h] = \frac{\mathrm{i}}{2}\int \mathrm{d}x\, \mathrm{d}y\, h(x)G_0(x-y)h(y) \tag{2.5}$$

to get

$$Z_0'[h, j] = Z_0[h]\int \mathscr{D}[\chi]\exp\left(\mathrm{i}\mathfrak{A}[\chi; h] + \mathrm{i}\int j\chi\right) \tag{2.6}$$

where

$$\mathfrak{A}[\chi, h] = \mathfrak{A}_0[\chi, h] - \frac{\mathrm{i}}{2}\int \mathrm{d}x\, \mathrm{d}y\, h(x)G_0(x-y)h(y). \tag{2.7}$$

From (1.4) we see that $\mathfrak{A}[\chi, h]$ can be expressed as

$$\mathfrak{A}[\chi, h] = -\tfrac{1}{2}\eta G_0(0)\int \mathrm{d}x\, \chi^2(x) - \tfrac{1}{2}\eta\int \mathrm{d}x\, \chi^2(x)\Phi^2(x)$$
$$+\tfrac{1}{2}\sum_{n=2}^{\infty}\eta^n\int \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n V^{(2n)}(x_1 \ldots x_n; h)\chi^2(x_1)\ldots\chi^2(x_n). \tag{2.8}$$

In equation (2.8)

$$\Phi(x) = \mathrm{i}\int \mathrm{d}y\, G_0(x-y)h(y) \tag{2.9}$$

and

$$V^{(2n)}(x_1 \ldots x_n) = V_0^{(2n)}(x_1 \ldots x_n) + V_1^{(2n)}(x_1 \ldots x_n; h) \tag{2.10}$$

where

$$V_0^{(2n)}(x_1 \ldots x_n) = \frac{(-\mathrm{i})^{n+1}}{n} G_0(x_1-x_2)G_0(x_2-x_3)\ldots G_0(x_n-x_1) \tag{2.11}$$

is h independent, and

$$V_1^{(2n)}(x_1 \ldots x_n) = -(-\mathrm{i})^{n-1}\Phi(x_1)G_0(x_1-x_2)\ldots G_0(x_{n-1}-x_n)\Phi(x_n) \tag{2.12}$$

is h dependent through $\Phi(x)$. In figure 1 $V_0^{(2n)}$ and $V_1^{(2n)}$ are represented diagramatically by a closed φ n-polygon and $(n+1)$-link chain terminated by h respectively.

What really interests us is the generating functional $W_0'[h]$ for *connected* pseudo-free Green functions obtained from

$$\mathrm{i}W_0'[h, j] \equiv \ln Z_0'[h, j] \tag{2.13}$$

on setting $j=0$. From its definition, we see that

$$W_0'[h] = W_0[h] + \Sigma[h] \tag{2.14}$$

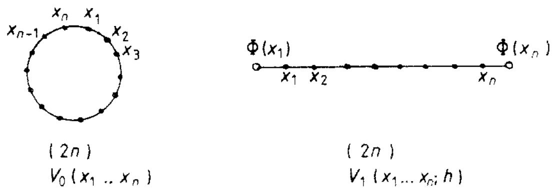


Figure 1. The non-local interaction vertices $V_0^{(2n)}$ and $V_1^{(2n)}$ of the effective action $\mathfrak{A}[\chi; h]$ of equation (2.8).

where $\Sigma[h]$ is the sum of all *connected* vacuum-to-vacuum diagrams constructed from a χ theory with action $\mathfrak{A}[\chi; h]$ of equation (2.8). That is, from a theory with propagator

$$\Delta_0(x-y) = -\frac{\mathrm{i}}{\eta G_0(0)}\delta(x-y) \tag{2.15}$$

h-dependent mass insertion (2-point function)

$$M^2(x) = \eta\Phi^2(x) \tag{2.16}$$

and (partially) h-dependent non-local $2n$-point vertices $-\frac{1}{2}\eta^n V^{(2n)}$ (as given by figure 1).

For reasons that will become clearer later, the most relevant renormalisation to consider is that of the parameter m_0^2. To see this it is sufficient to consider the *two-point* function

$$G_2(x-y) = \frac{\delta^2 W}{\mathrm{i}^2\delta h(x)\,\delta h(y)}\bigg|_{h=0} = G_0(x-y) + \frac{\delta^2\Sigma}{\mathrm{i}^2\,\delta h(x)\delta h(y)}\bigg|_{h=0} \tag{2.17}$$

The first term in (2.17) is the free-field solution. If we examine the second term we see that only the two following types of diagram can contribute.

(i) Diagrams with *no* M^2 insertions. In this case all except one of the vertices $V^{(2n)}$ are replaced by $V_0^{(2n)}$. The remaining vertex is replaced by $V_1^{(2n)}$ which is then differentiated twice to remove its h. This is done in all possible ways.

(ii) Diagrams with *one* M^2 insertion which is doubly differentiated. All vertices $V^{(2n)}$ are then replaced by $V_0^{(2n)}$.

Table 1 displays these diagrams at one-loop, two-loop, and three-loop levels. For convenience we take

$$\eta G_0(0) = 1 \tag{2.18}$$

in these diagrams to simplify calculations. With this choice, the propagator Δ_0 is a δ function with coefficient unity, and $V^{(2n)}$ has the coupling strength $G_0^{-n}(0)$ associated with it (and M^2 the coefficient $G_0^{-1}(0)$).

Individual entries in table 1 are interpreted as follows. The second column displays a particular connected vacuum-to-vacuum diagram constructed from the above propagator and vertices. In the third column are displayed those contributions to $G(x-y)$ obtained from this vacuum diagram. The broken lines represent the δ function Δ, the full lines the G_0. On contracting the Δ we obtain the diagrams in the fourth column. To determine the nature of the singularities associated with each diagram we note that each closed broken loop in the uncontracted diagrams gives one factor of $\delta(0)$, and each vertex gives a factor of $G_0^{-1}(0)$. We then have to multiply these singularities by the singularities ($G_0(0)$, B, etc) associated with the contracted diagram.

Table 1. The zero-, one-, two- and three-loop contributions to the two-point function $G_2(xy)$.

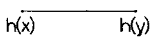
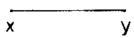
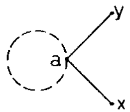
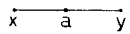
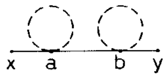
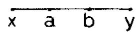
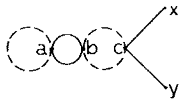
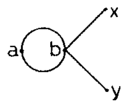
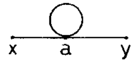
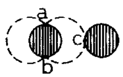
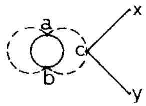
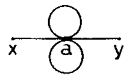
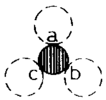
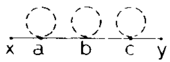
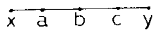
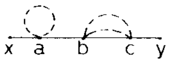
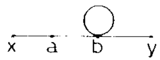
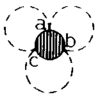
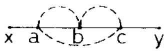
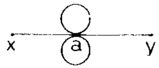
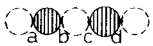
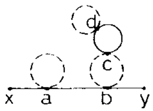
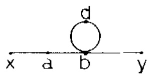
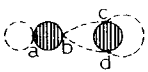
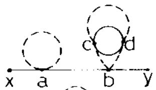
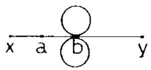
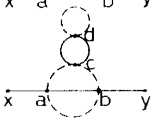
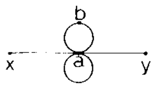
Number of loops	Relevant diagrams of $W[h]$	$G_2(x-y)=\left.\frac{\delta^2 W}{\delta h(x)\delta h(y)}\right\|_{h=0}$ Uncontracted	Contracted	Singularity
0				1
1				$\frac{\delta(0)}{G_0(0)}$
2(*a*)				$\left(\frac{\delta(0)}{G_0(0)}\right)^2$
2(*b*)				$\frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2$
2(*c*)				$\frac{\delta(0)}{G_0(0)}$
2(*d*)				$\frac{\delta(0)}{G_0(0)}$

Table 1. (continued)

Number of loops	Relevant diagrams of $W[h]$	$G_2(x-y)=\left.\frac{\delta^2 W}{\delta h(x)\delta h(y)}\right\|_{h=0}$ Uncontracted	Contracted	Singularity
$3(a)$				$\left(\frac{\delta(0)}{G_0(0)}\right)^3$
$3(b)$				$\left(\frac{\delta(0)}{G_0(0)}\right)^2$
$3(c)$				$\frac{\delta(0)}{G_0(0)}$
$3(d)$				$\frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2$
$3(e)$				$\left(\frac{\delta(0)}{G_0(0)}\right)^2$
				$\frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2$

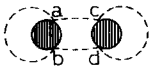
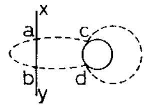
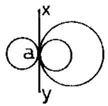
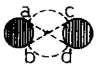
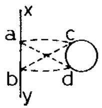
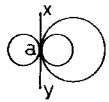
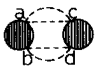
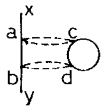
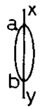
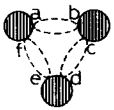
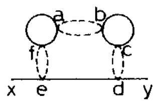
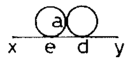
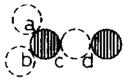
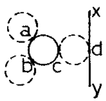
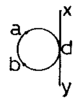
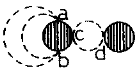
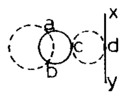
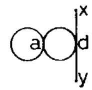
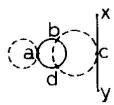
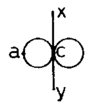
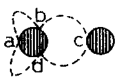
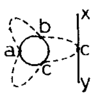
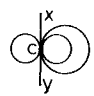
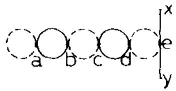
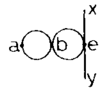
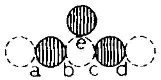
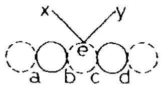
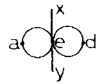
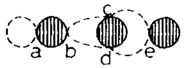
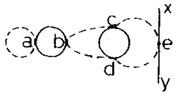
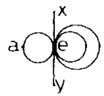
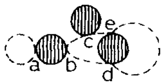
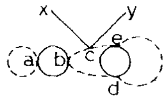
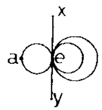
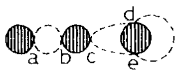
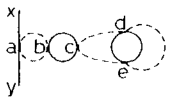
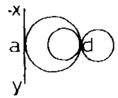
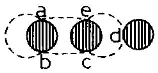
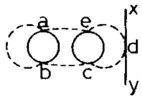
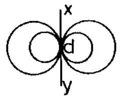
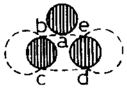
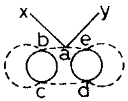
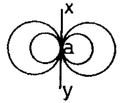
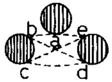
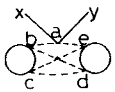
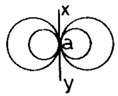
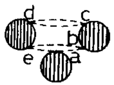
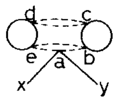
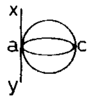
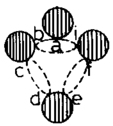
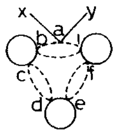
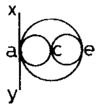
3(f)				$\dfrac{\delta(0)}{G_0(0)}$
3(g)				$\dfrac{\delta(0)}{G_0(0)}$
3(h)				$\dfrac{D}{(G_0(0))^2}\left(\dfrac{\delta(0)}{G_0(0)}\right)^2$
3(i)				$\dfrac{E}{(G_0(0))^3}\left(\dfrac{\delta(0)}{G_0(0)}\right)^3$
3(j)				$\dfrac{L_3}{G_0(0)}\left(\dfrac{\delta(0)}{G_0(0)}\right)^3$
3(k)				$\dfrac{L_2}{G_0(0)}\left(\dfrac{\delta(0)}{G_0(0)}\right)^2$
3(l)				$\dfrac{L_2}{G_0(0)}\left(\dfrac{\delta(0)}{G_0(0)}\right)^2$
3(m)				$\dfrac{\delta(0)}{G_0(0)}$

Table 1. (continued)

Number of loops	Relevant diagrams of $W[h]$	$G_2(x-y)=\left.\frac{\delta^2 W}{\delta h(x)\delta h(y)}\right\vert_{h=0}$ Uncontracted	Contracted	Singularity
$(3n)$				$\left(\frac{L_2}{G_0(0)}\right)^2\left(\frac{\delta(0)}{G_0(0)}\right)^3$
$3(o)$				$\left(\frac{L_2}{G_0(0)}\right)^2\left(\frac{\delta(0)}{G_0(0)}\right)^3$
$3(p)$				$\frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2$
$3(q)$				$\frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2$
$3(r)$				$\frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2$

3(s)				$\dfrac{\delta(0)}{G_0(0)}$
3(t)				$\dfrac{\delta(0)}{G_0(0)}$
3(u)				$\dfrac{\delta(0)}{G_0(0)}$
3(v)				$\dfrac{B}{(G_0(0))^3}\left(\dfrac{\delta(0)}{G_0(0)}\right)^2$
3(w)				$\dfrac{F}{(G_0(0))^4}\left(\dfrac{\delta(0)}{G_0(0)}\right)^3$

For example, in figure 3(c) of the table we have one $\delta(0)$ and four factors of $G_0^{-1}(0)$ from the uncontracted diagram. Since the contracted diagram contains $G_0^3(0)$ the resultant singularity is $\delta(0)/G_0(0)$. The notation is that

$$L_n = \int \prod_{r=1}^{n-1} \mathrm{d}x_r\, G_0(x-x_1)G_0(x_1-x_2)\dots G_0(x_{n-1}-x) \tag{2.19}$$

is an n-link loop fixed at x. Other symbols (D, E, etc) have a meaning that can be read off from the corresponding diagram in the fourth column.

The last column contains the nature of the formal singularity associated with the diagram. On examination, we see that each singularity occurs at (infinitely) many-loop levels. For example, $\delta(0)/G_0(0)$ will arise from the n-petal 'flower' (interpreting diagrams 2(c), 3(c), 3(f), 3(s) as one-, two-, three- and four-petal 'flowers') for arbitrary n. This indicates the effective non-polynomial nature of the 'hard-core' scale-invariant measure. However, because of our inability to calculate the coefficient of this singularity at each loop level (for which it will occur several times) it is sufficient for the moment just to catalogue its presence.

In order to understand how to handle such formally meaningless singularities we examine other circumstances in which a similar problem arises.

The best understood is the $1/N$ expansion of O(N)-invariant linear and nonlinear σ models of scalar fields (belonging to the vector representation). For these models the leading term in the $1/N$ expansion is the sum of all 'cactus' diagrams (Schnitzer 1974) i.e. diagrams constructed entirely from $G_0(0)$ and L_n 'petals'. The formal similarity extends even to the fact that the coupling strength λ_0 (analogous to η above) associated with these 'petals' is formally expressible as the inverse of singular distributions.

If this $1/N$ expansion has taught us anything, it is that the essential physics implied by it (e.g. dynamical changes of symmetry) arises from these most singular diagrams. Any attempt to eliminate them by normal ordering would be disastrous†. Similar arguments can be made for mean-field diagrammatic expansions (Bender *et al* 1977). However, as a consolation we have learnt to expect these singularities, which get progressively worse in loop expansions, to sum to something simpler.

Yet further support comes from orthodox non-polynomial field theories, where we find (Salam 1971) that attempts to normal-order potentially give rise to inconsistencies.

Bearing this in mind we conclude this discussion of W_2 by singling out those most singular diagrams which make contributions only to point-mass insertions. If m^2 is the total mass given by such insertions, it is formally expressible in terms of m_0^2 as

$$m^2 = m_0^2 + a\frac{\delta(0)}{G_0(0)} + \frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2\left(b_1 + b_2\frac{L_2}{G_0(0)}\cdot\frac{\delta(0)}{G_0(0)} + \dots\right)$$
$$+\frac{c_1L_3}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^3 + \frac{d_1L_4}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^4 + \dots. \tag{2.20}$$

There will also be cross terms in L_2L_3 etc, present at a higher loop level. In (2.20) the coefficients $a, b_1, b_2, \dots, c_1, \dots, d_1, \dots$ are uncalculable and $G_0(0)$ depends formally on m_0^2.

† This has been shown empirically for general $1/N$ expansions. In § 6 we shall specifically calculate the large-N limit of the O(N)-invariant scalar pseudo-free theory.

As we might have expected, the series (2.20) can be formally summed, and we shall show how to do so later. What is important for the moment is that from the above discussion we believe that the series (2.20) is the one we have to consider.

We conclude this section with the recognition that any complete description of the pseudo-free theory must go beyond the two-point function. In table 2 we have displayed some of the diagrams that contribute to the four-point connected Green function $W_4(x_1x_2x_3x_4)$. We shall have a few comments to make on this later.

3. The pseudo-free branching equations

After the full frontal diagrammatic assault on the generating functional $Z_0'[h]$ of (1.1) we shall now be a little more subtle. We have at least learnt not to attempt any normal ordering. Our approach will be to attempt a solution based on the branching equations for $Z_0'[h, j]$ of (2.1).

Accepting (2.1) at its face value, from the translation invariance of $\mathscr{D}[\varphi]$, $\mathscr{D}[\chi]$ it follows that

$$\left(h(x)-K_x\frac{\delta}{\mathrm{i}\,\delta h(x)}-\eta\frac{\delta^3}{\mathrm{i}^3\,\delta h(x)\,\delta j(x)^2}\right)Z_0'[h,j]=0 \tag{3.1}$$

and

$$\left(j(x)-\eta\frac{\delta^3}{\mathrm{i}^3\,\delta j(x)\,\delta h(x)^2}\right)Z_0'[h,j]=0 \tag{3.2}$$

where $K_x=\Box_x+m_0^2$.

Equation (3.2) is a constraint equation, unaffected in form by the inclusion of interactions, whereas (3.1) is more dynamical.

The branching equations that follow from (3.1) are easily obtained. If the unconnected Green functions are defined by

$$G_{m,n}(x_1\ldots x_m;y_1\ldots y_n)=\frac{1}{\mathrm{i}^{m+n}}\left.\frac{\delta^{m+n}Z_0'[h,j]}{\delta h(x_1)\ldots\delta h(x_m)\delta j(y_1)\ldots\delta j(y_n)}\right|_{h=j=0} \tag{3.3}$$

($Z_0'[0,0]=1$) the constraint equation (3.2) gives

$$\mathrm{i}\delta(x-y)+\eta G_{2,2}(xx;xy)=0 \tag{3.4a}$$

$$\mathrm{i}\,\delta(x-y)G_{p,0}(x_1\ldots x_p)+\eta G_{p+2,2}(xxx_1\ldots x_p;xy)=0 \qquad p\geq 2 \tag{3.4b}$$

and more generally

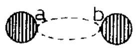
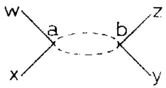
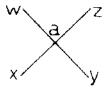
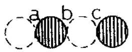
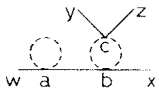
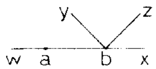
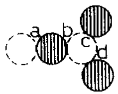
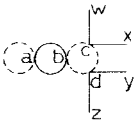
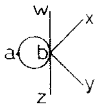
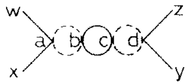
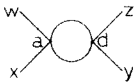
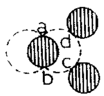
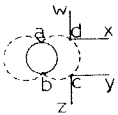
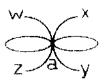
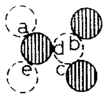
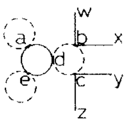
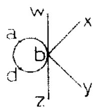
$$\mathrm{i}\sum_{r=1}^{q+1}\delta(x-x_r)G_{p,q}(x_1\ldots x_p;y_1\ldots\hat{y}_r\ldots y_{q+1})$$
$$+\eta G_{p+2,q+2}(xxx_1\ldots x_p;xy_1\ldots y_q)=0 \qquad q\geq 2. \tag{3.4c}$$

Since $G_{2p+1,q}$, $G_{p,2q+1}$ decouple in all equations from $G_{2p,2q}$ we have set them to zero. Equations (3.4) have been displayed in I. We note that (3.4b) can be written

$$G_{p,0}(x_1\ldots x_p)G_{2,2}(xx;xy)=G_{p+2,2}(xxx_1\ldots x_p;xy) \tag{3.5}$$

showing a high degree of factorisation (figure 2).

Table 2. Examples from the one-, two- and three-loop contributions to the four-point connected Green function $W_4(wxyz)$.

Number of loops	Relevant diagrams of $W[h]$	$W_4(wxyz)=\left.\dfrac{\delta^4 W[h]}{\delta h(w)\delta h(x)\delta h(y)\delta h(x)}\right\vert_{h=0}$ Uncontracted	Contracted	Singularity
1				$\dfrac{\delta(0)}{(G_0(0))^2}$
2(*a*)				$\dfrac{\delta^2(0)}{G_0^3(0)}$
2(*b*)				$\dfrac{L_2}{(G_0(0))^2}\left(\dfrac{\delta(0)}{G_0(0)}\right)^2$
2(*c*)				$\dfrac{E_2}{(G_0(0))^2}\left(\dfrac{\delta(0)}{G_0(0)}\right)^2$
2(*d*)				$\dfrac{\delta(0)}{G_0^2(0)}$
3(*a*)				$\dfrac{L_3}{G_0^2(0)}\left(\dfrac{\delta(0)}{G_0(0)}\right)^3$

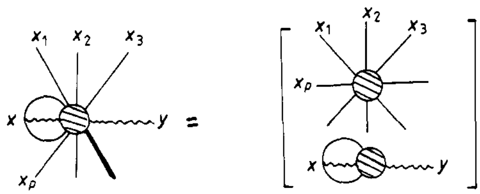


Figure 2. The exact factorisation equation (3.5). Full lines refer to the φ field, wavy lines to the χ field. Circles denote unconnected Green functions and broken lines represent delta functions.

The dynamical equation (3.1) gives

$$\delta(x-y)+K_xG_{2,0}(xy)-\eta G_{2,2}(xy;xx)=0 \tag{3.6a}$$

$$\mathrm{i}\sum_{r=1}^{m}\delta(x-x_r)G_{m-1,0}(x_1\ldots\hat{x}_r\ldots x_m)+K_xG_{m+1,0}(xx_1\ldots x_m)$$
$$+\eta G_{m+1,2}(xx_1\ldots x_m;xx)=0 \qquad m\geqslant 2 \tag{3.6b}$$

$$\mathrm{i}\sum_{r=1}^{m}\delta(x-x_r)G_{m-1,q}(x_1\ldots\hat{x}_r\ldots x_m;y_1\ldots y_q)+K_xG_{m+1,q}(xx_1\ldots x_m;y_1\ldots y_q)$$
$$+\eta G_{m+1,q+2}(xx_1\ldots x_m;xxy_1\ldots y_q)=0 \qquad m,q\geqslant 2 \tag{3.6c}$$

displayed in I.

It is useful to compare these equations with the *subtracted* scale-covariant equations obtained from (1.2). These are (Klauder 1977)

$$\left[h(x)\frac{\delta}{\mathrm{i}\delta h(x)}-:\frac{\delta}{\mathrm{i}\delta h(x)}K_x\frac{\delta}{\mathrm{i}\delta h(x)}:\right]Z_0'[h]=0 \tag{3.7}$$

with

$$:\frac{\delta^p}{\delta h(x)^p}:Z_0'=\frac{\delta^p}{\delta h(x)^p}Z_0'-\left.\frac{\delta^pZ_0'}{\delta h(x)^p}\right|_{h=0}Z_0'. \tag{3.8}$$

The subtraction procedure (3.8) is motivated by the lattice formulation of the independent-value model (IVM), and was discussed in II.

We see that, to make a comparison, we replace equation (3.1) by the higher-order equation (obtained from (3.1) and (3.2))

$$0=\left[h(x)\frac{\delta}{\mathrm{i}\delta h(y)}-j(x)\frac{\delta}{\mathrm{i}\delta j(y)}-\frac{\delta}{\mathrm{i}\delta h(y)}K_x\frac{\delta}{\mathrm{i}\delta h(x)}\right.$$
$$\left.-\eta\left(\frac{\delta^4}{\delta h(x)\delta h(y)\delta j(x)^2}-\frac{\delta^4}{\delta h(x)^2\delta j(x)\delta j(y)}\right)\right]Z_0'=0. \tag{3.9}$$

On setting $x=y$ and $j=0$ equation (3.9) becomes identical to (3.7) but for the absence of any subtraction. We note that, since $G_{2,1}=G_{1,2}=0$ equations (3.1) and (3.2) are immune to subtraction. In order to restore parity between the augmented equations and (3.7) either (3.4a) or (3.6a) would have to be rejected†. As we have

† In II we argued that (3.4a), at least, is sound.

no compelling reason to jettison either we shall examine the consequence of assuming (3.9) to be correct. That is, the augmented equations contain more information than the subtracted scale-covariant equations.

For the remainder of this section we shall extract this additional information. Using the simplifying notation G_p for $G_{p,0}$ we see, on subtracting (3.4a) from (3.6a), that

$$\lim_{x\to y} K_x G_2(x-y) = 0. \tag{3.10}$$

This equation is the essential new feature of the augmented equations. To see what it means we assume that $G_2(x-y)$ has the spectral representation ($\text{đ}k \equiv \mathrm{d}^n k/(2\pi)^n$)

$$G_2(x-y) = -\mathrm{i}\int \text{đ}k\, \exp[\mathrm{i}k(x-y)] \int \mathrm{d}\sigma \frac{\rho(\sigma)}{k^2-\sigma^2} \tag{3.11}$$

$$= \int \mathrm{d}\sigma\, G_0(x-y;\sigma)\rho(\sigma). \tag{3.12}$$

Equation (3.10) can then be formally expressed as

$$m_0^2 = \bar{m}^2 - \delta(0)/\mathrm{i}G_2(0) \tag{3.13}$$

where

$$G_2(0)\bar{m}^2 = \int \mathrm{d}\sigma\, \sigma\rho(\sigma)G_0(0;\sigma). \tag{3.14}$$

How should we interpret (3.13)? Firstly, there is a sense in which it is a statement of mass renormalisation. Suppose

$$\rho(\sigma) = \delta(\sigma - m^2) + c(\sigma) \tag{3.15}$$

where m is the renormalised mass of the theory, and c characterises the continuum contribution. Equation (3.13) can then be re-expressed as

$$m^2 = m_0^2 + \delta(0)/\mathrm{i}G_2(0) - \bar{c} \tag{3.16}$$

where $\bar{c}$ is the continuum contribution. Since $\mathrm{i}G_2(0)$ can, in turn, be formally expressed as

$$\mathrm{i}G_2(0) = \mathrm{i}G_0(0;m^2) + \mathrm{i}G_2(\text{continuum}) \tag{3.17}$$

with

$$\mathrm{i}G_0(0;m^2) = -\int \frac{\text{đ}k_E}{k^2+m^2} \tag{3.18}$$

equation (3.16) is, in part, a self-consistent expression of the mass renormalisation that is required.

The second property of (3.13) is that, since the second term is infinitely negative, our expectation is that m_0^2 is infinitely negative. We do not consider this a problem. For example, let us consider the strong coupling expansion based on the IVM as developed in Kovesi-Domokos (1976). Even if we are uneasy about the possible over regularisation used by Kovesi-Domokos, a formal mass renormalisation of the type

$$m^2 = \tfrac{1}{4}b\delta(0)m_0^2 + \int k^2\, \text{đ}k \tag{3.19}$$

(m_0 is the unrenormalised mass) is at least part of the picture. Equation (3.19), with some similarities to (3.16), also requires that m_0^2 be infinitely negative.

To allay any remaining doubts about the validity of (3.10) and its consequences, we shall now argue that the diagrammatic expansion of § 2 is no more than a description of (3.16) and hence of (3.10).

4. The diagrams reconsidered

We wish to reconcile the qualitative expansion (2.20) to the self-consistent mass renormalisation of (3.16). The first thing we note is that the series (2.20) represents only the most singular part of the self-mass, with no continuum contributions. This corresponds to taking

$$\rho(\sigma) \approx \delta(\sigma - m^2) \tag{4.1}$$

in (3.15), whence equation (3.16) becomes the self-consistent relation

$$m^2 = m_0^2 + \delta(0)/\mathrm{i}G_0(0; m^2). \tag{4.2}$$

Since $G_0(0)$ in equation (2.20) is shorthand for $G_0(0; m_0^2)$ we need to express $G_0(0; m^2)$ in terms of $G_0(0; m_0^2)$. If

$$\delta m^2 = m^2 - m_0^2 \tag{4.3}$$

we have

$$G_0(0; m^2) = G_0(0; m_0^2) + \mathrm{i}\delta m^2 L_2 + (\mathrm{i}\delta m^2)^2 L_3 + \ldots + \ldots (\mathrm{i}\delta m^2)^{p-1} L_p + \ldots \tag{4.4}$$

with L_n as given in (2.19).

If we now use the fact that δm^2 is itself given by (4.2) as

$$\delta m^2 = \delta(0)/\mathrm{i}G_0(0; m^2) \tag{4.5}$$

we see that $G_0(0; m^2)$ can be expanded in continued fractions as

$$\begin{aligned} G_0(0; m^2) = G_0(0) &+ \frac{\delta(0)}{G_0(0)} L_2 \left[1 + \frac{\delta(0)}{G_0(0)} \frac{L_2}{[G_0(0) + \ldots]}\right]^{-1} \\ &+ \left(\frac{\delta(0)}{G_0(0)}\right)^2 L_3 \left[1 + \frac{\delta(0)}{G_0(0)} \frac{L_2}{[G_2(0) + \ldots]}\right]^{-1} + \left(\frac{\delta(0)}{G_0(0)}\right)^3 L_4 [1 + \ldots]^{-1} + \ldots \end{aligned} \tag{4.6}$$

where as before $G_0(0) \equiv G_0(0; m_0^2)$.

Expanding $G_0(0; m^2)^{-1}$ about $G_0^{-1}(0)$ then gives

$$\begin{aligned} \delta m^2 &= m^2 - m_0^2 \\ &= \frac{\delta(0)}{\mathrm{i}G_0(0)} + \frac{L_2}{\mathrm{i}G_0(0)} \left(\frac{\delta(0)}{\mathrm{i}G_0(0)}\right)^2 \\ &\quad + \frac{2L_2^2}{(\mathrm{i}G_0(0))^2} \left(\frac{\delta(0)}{\mathrm{i}G_0(0)}\right)^3 + \ldots - \frac{L_3}{\mathrm{i}G_0(0)} \left(\frac{\delta(0)}{\mathrm{i}G_0(0)}\right)^3 + \ldots. \end{aligned} \tag{4.7}$$

As required, the singularities arising in (4.7) are just those arising in (2.20).

We now understand the diagrammatic expansion of G_2 in § 2 as serving two related purposes. On the one hand, it expresses the nature of the mass renormalisation. On the other, by plausibly reproducing (4.2) and presumably (if the non-leading singularities could be handled) satisfying (3.16) it guarantees that equation (3.10) is satisfied. Since the diagrams of table 1 correspond to an effective *non-normal ordered* theory, and equation (3.10) to a *non-subtracted* theory we have identified the subtraction procedure with a normal ordering.

We have a little insight into a further puzzle, that of the *homogeneity* of the scale-covariant branching equations (3.7) (with or without subtractions). On taking the operator-product expansion implied by these into account we get formally *linear homogeneous* equations for the *connected* Green functions W_n. The W_n obtained from these equations thus have an undetermined scale b (Klauder 1978, 1979). One hope expressed for the augmented formalism was that, by virtue of the *inhomogeneous* terms in the branching equations (3.4) and the expected *nonlinearity* of the connected Green function equations, this scale could be determined. The case of $G_2(x)$ presented above shows that the diagrams for $G_2(x)$, rather than determining it in terms of the free-field propagator $G_0(x)$, merely conspire to guarantee that the homogeneous equation (3.10) is satisfied.

As a final consistency check, let us see what happens if we replace $K_x = \Box_x + m_0^2$ by m_0^2 in the previous analysis. On the one hand, dropping the kinetic term suggests that we should retrieve the IVM. On the other, we know that the IVM crucially requires the subtraction procedure that we have now rejected. These opposing expectations are reconciled in the following way. Firstly, the only singularity that arises is $\delta(0)$, with no analogue of 'continuum' contributions. Equation (4.2) now becomes

$$m^2 = m_0^2 + m^2 \tag{4.8}$$

with solution $m_0^2 = 0$†. Thus equation (3.10) is still satisfied, albeit trivially. More to the point, massless pseudo-free IVM does not exist, resolving the contradiction.

5. Towards a solution of the augmented equations

We shall further confirm the validity of the constraint equation (3.10) by presenting a solution preserving the leading-order singularities of branching equations (3.1) and (3.2) (or equivalently (3.4) and (3.6)).

We first consider the constraint equation (3.2) and its corresponding branching equations (3.4). We shall look for the simplest solutions to the most singular parts of these equations and check their consistency with the dynamical equations afterwards.

We note immediately (equation (3.4*a*)) that the singularity arising from making two χ fields coincident is a δ function. This is compatible with χ being a purely auxiliary field, even after renormalisation, for which we would expect

$$G_{0,2}(x, y) \propto \delta(x-y). \tag{5.1}$$

Taking this to be the case (3.4*b*) becomes consistent if, as far as leading singularities are concerned,

$$G_{2,2}(xx; xy) \approx G_{2,0}(xx)G_{0,2}(xy). \tag{5.2}$$

† Since each contribution in table 1 to the single-mass insertion is proportional to m^2, an alternative solution would be to interpret (4.8) as $m^2 = m_0^2 + m_0^2 + \ldots + m_0^2 + \ldots$, that is m^2 infinite whereby the theory again ceases to exist.

A simple way to make equations (3.4b) identical to (3.4a) is to further assume that (compatible with (5.2))

$$G_{p+2,2}(xxx_1\ldots x_p;xy)\approx G_{p+2,0}(xxx_1\ldots x_p)G_{0,2}(xy) \tag{5.3}$$

and

$$G_{p+2,0}(xxx_1\ldots x_p)\approx G_{2,0}(xx)G_{p,0}(x_1\ldots x_p) \tag{5.4}$$

are satisfied (again at the level of leading singularities). This is displayed in figure 3.

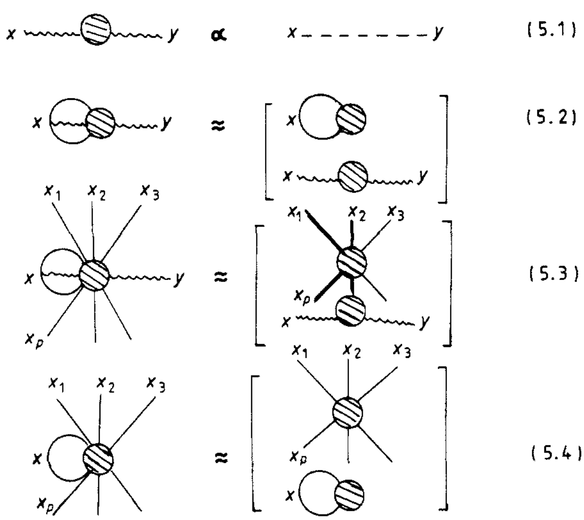


Figure 3. The approximate factorisation equations (5.1) to (5.4). Circles denote unconnected Green functions.

Klauder's augmented equations for $G_{p,0}$ (the quantities of interest) only directly involve $G_{p,q}$ for $q\geqslant 2$. It is plausible that assumptions (5.1)–(5.4) alone, when used in equations (3.6), provide a basis for constructing solutions for $G_{p,0}$. We stress that these assumptions, compatible with equations (3.4), are *not* tailored to satisfy the crucial equation (3.10), which requires (3.6a) in addition.

However, we observe that all these assumptions (and more) are a consequence of the single assumption of separability

$$Z_0'[h,j]\approx H[h]J[j] \tag{5.5}$$

where the equality is to be interpreted as applying *only* to the most singular contributions.

Inserting (5.5), motivated by the constraint equations (3.4), into the constraint equation (3.2) gives

$$[j(x)J]H\approx\eta\left(\frac{\delta J}{\mathrm{i}\delta j(x)}\right)\left(\frac{\delta^2 H}{\mathrm{i}^2\delta h(x)^2}\right) \tag{5.6}$$

with solution

$$\frac{\delta^2 H}{\mathrm{i}^2\delta h(x)^2}\approx G_{2,0}(xx)H\equiv G_2(0)H \tag{5.7}$$

and

$$\eta G_2(0)\frac{\delta J}{\mathrm{i}\delta j(x)}\approx j(x)J. \tag{5.8}$$

Since η is an arbitrary scale factor, we choose it to satisfy†

$$\eta G_2(0)=1. \tag{5.9}$$

Equation (5.8) now shows that as far as leading singularities *alone* are concerned J describes a 'free-field' theory with two-point function

$$\Delta_2(x-y)\equiv G_{0,2}(x-y)\approx-\mathrm{i}\delta(x-y). \tag{5.10}$$

In its own way, equation (5.7), giving (5.4) directly, states that H also describes a 'free theory' (to leading singularities).

To check the consistency of this we insert (5.5) into the dynamical equation (3.1) to obtain

$$\left[h(x)-K_x\frac{\delta}{\mathrm{i}\delta h(x)}\right]H\approx\eta\frac{\delta H}{\mathrm{i}\delta h(x)}\frac{\delta^2 J}{\mathrm{i}^2\delta j(x)^2}\approx\frac{\delta H}{\mathrm{i}\delta h(x)}\left[\frac{\delta(0)}{\mathrm{i}G_2(0)}+\frac{j(x)^2}{G_2(0)}\right]. \tag{5.11}$$

Rewriting this as

$$\left[h(x)-\left(K_x+\frac{\delta(0)}{\mathrm{i}G_2(0)}\right)\frac{\delta}{\mathrm{i}\delta h(x)}\right]H\approx\frac{\delta H}{\mathrm{i}\delta h(x)}\cdot\frac{j(x)^2}{G_2(0)} \tag{5.12}$$

we assume that the superficial inconsistency of a j-dependent right-hand side is interpreted by making the infinite $G_2(0)$ take the right-hand side to zero. We thus end up with the equation

$$\left[h(x)-\left(K_x+\frac{\delta(0)}{\mathrm{i}G_2(0)}\right)\frac{\delta}{\mathrm{i}\delta h(x)}\right]H\approx 0 \tag{5.13}$$

showing a field with self-consistent additive mass renormalisation of m_0^2 to

$$m^2=m_0^2+\delta(0)/\mathrm{i}G_2(0). \tag{5.14}$$

At the level of leading singularities this is just the result (4.2) of the previous section.

We accept that it is easy to be deceived by heuristic equations of the type shown above. To justify the step from (5.12) to (5.13) we really need to study the branching equations (3.4) and (3.6) in more detail. In the appendix we show how consistency is achieved.

So far everything hangs together at the level of leading singularities, at which level the connected (truncated) Green functions play no role, because of the approximate 'free-field' solution. On examining the contributions to W_4 of table 2 we are unable to see how the contributions may sum. Nevertheless, we have no reason to believe that they sum to a vanishing contribution except insofar as $\delta(0)/G_0(0)^2$ vanishes.

We shall return to this in a later paper.

6. The large-N limit of the O(N) pseudo-free theory

In our discussion so far, the equation (4.2) has a pivotal position. It serves the three roles of describing the nature of mass renormalisation, implying the existence of the

† In § 2 we chose $\eta G_0(0)=1$.

branching equation (3.10), and reflecting the consistent separability of the most singular contributions within the general branching equations.

However, it is an *approximate* equation, obtained by retaining only the most singular contributions (the point-mass insertions) induced by the hard-core effect of the scale-invariant measure. We shall now show that this equation (4.2) becomes *exact* in the large-N limit of the O(N)-invariant scale-covariant pseudo-free scalar theory. This serves to reinforce further our belief in the results presented so far.

Consider this O(N)-invariant scalar theory, with N scalar fields φ_i ($i = 1, 2, \ldots, N$) and generating functional

$$Z'_0[\boldsymbol{h}] = \int \mathscr{D}'[\boldsymbol{\varphi}] \exp \mathrm{i} \int \mathrm{d}x\, [\tfrac{1}{2}(\partial_\mu \boldsymbol{\varphi})^2 - \tfrac{1}{2} m_0^2 \boldsymbol{\varphi}^2 + \boldsymbol{h} \cdot \boldsymbol{\varphi}] \tag{6.1}$$

where the O(N)-invariant measure $\mathscr{D}'[\boldsymbol{\varphi}]$ is also invariant under scale transformations, i.e.

$$\mathscr{D}'[\Lambda \boldsymbol{\varphi}] = \mathscr{D}'[\boldsymbol{\varphi}]. \tag{6.2}$$

To preserve (6.2) we generalise the auxiliary formalism (1.1) to

$$Z'_0[\boldsymbol{h}] = \int \mathscr{D}[\boldsymbol{\varphi}] \mathscr{D}[\boldsymbol{\chi}] \exp \mathrm{i} \int \mathrm{d}x [\tfrac{1}{2}(\partial_\mu \boldsymbol{\varphi})^2 - \tfrac{1}{2} m_0^2 \boldsymbol{\varphi}^2 - \tfrac{1}{2} \eta N^{-1} \boldsymbol{\chi}^2 \boldsymbol{\varphi}^2 + \boldsymbol{h} \cdot \boldsymbol{\varphi}] \tag{6.3}$$

where the O(N)-invariant measures $\mathscr{D}[\boldsymbol{\varphi}]$, $\mathscr{D}[\boldsymbol{\chi}]$ are now translation invariant. We assume that the functional integral is dominated by regions for which $\boldsymbol{\varphi}^2$, $\boldsymbol{\chi}^2$ are of order N, whence the explicit N dependence of the cross term $\boldsymbol{\chi}^2 \boldsymbol{\varphi}^2$ (η is O(1)). In order to make the N dependence even more explicit we rewrite $Z'_0[\boldsymbol{h}]$ as

$$Z'_0[\boldsymbol{h}] = \int \mathscr{D}[\boldsymbol{\varphi}] \mathscr{D}[\boldsymbol{\chi}] \mathscr{D}[\sigma] [\delta(\boldsymbol{\chi}^2 - N\sigma)] \exp \mathrm{i} \int \mathrm{d}x [\tfrac{1}{2}(\partial_\mu \boldsymbol{\varphi})^2 - \tfrac{1}{2}(m_0^2 + \eta\sigma) \boldsymbol{\varphi}^2 + \boldsymbol{h} \cdot \boldsymbol{\varphi}] \tag{6.4}$$

$$= \int \mathscr{D}[\boldsymbol{\varphi}] \mathscr{D}[\boldsymbol{\chi}] \mathscr{D}[\sigma] \mathscr{D}[\alpha] \exp \mathrm{i} \int \mathrm{d}x [\tfrac{1}{2}(\partial_\mu \boldsymbol{\varphi})^2 - \tfrac{1}{2}(m_0^2 + \eta\sigma) \boldsymbol{\varphi}^2 + \tfrac{1}{2} \alpha (\boldsymbol{\chi}^2 - N\sigma) + \boldsymbol{h} \cdot \boldsymbol{\varphi}] \tag{6.5}$$

where σ, α are O(1).

Performing the $\boldsymbol{\varphi}$ and $\boldsymbol{\chi}$ integrations gives

$$Z'_0[\boldsymbol{h}] = \int \mathscr{D}[\sigma] \mathscr{D}[\alpha] \exp \mathrm{i} N A[\sigma, \alpha; \boldsymbol{h}] \tag{6.6}$$

where

$$A = \frac{\mathrm{i}}{2} \mathrm{Tr} \ln \alpha (\Box + m_0^2 + \eta\sigma) - \frac{1}{2} \int \sigma\alpha + \frac{1}{2N} \int\int \boldsymbol{h} (\Box + m_0^2 + \eta\sigma)^{-1} \cdot \boldsymbol{h}. \tag{6.7}$$

Assuming that $\boldsymbol{h}^2$ is O(N), all terms in A are O(1).

This permits us to determine Z'_0 in the large-N limit by looking for the extremum of $A[\sigma, \alpha; \boldsymbol{h}]$, occurring at $\sigma_0[\boldsymbol{h}]$, $\alpha_0[\boldsymbol{h}]$. To leading order in N this gives

$$\mathrm{i} W'_0[\boldsymbol{h}] = \ln Z'_0[\boldsymbol{h}] = \mathrm{i} N A[\sigma_0, \alpha_0; \boldsymbol{h}]. \tag{6.8}$$

A more convenient quantitiy than W is the effective action Γ, the generating functional for the one φ irreducible Green functions. Defining the semiclassical fields $\boldsymbol{\varphi}$ by

$$\varphi_i = \delta W/\delta h_i \tag{6.9}$$

we have

$$\Gamma[\boldsymbol{\varphi}] = W[\boldsymbol{h}] - \int \boldsymbol{h} \cdot \boldsymbol{\varphi} \tag{6.10}$$

where W, $\boldsymbol{h}$ are expressed in terms of $\boldsymbol{\varphi}$ via (6.9).

From (6.7) we see that $\boldsymbol{\varphi}$ is defined by

$$(\Box + m_0^2 + \eta\alpha_0)\boldsymbol{\varphi} = \boldsymbol{h}. \tag{6.11}$$

This suggests that we introduce the generalised effective action

$$\Gamma[\boldsymbol{\varphi}, \alpha, \sigma] = -\frac{1}{2}\int \boldsymbol{\varphi} \cdot (\Box + m_0^2 + \eta\sigma)\boldsymbol{\varphi} - \frac{N}{2}\int \sigma\alpha + \tfrac{1}{2}\mathrm{i}N \operatorname{Tr} \ln \alpha(\Box + m_0^2 + \eta\sigma). \tag{6.12}$$

The effective action $\Gamma[\boldsymbol{\varphi}]$ of (6.10) is obtained from $\Gamma[\boldsymbol{\varphi}, \alpha, \sigma]$ as

$$\Gamma[\boldsymbol{\varphi}] = \Gamma[\boldsymbol{\varphi}, \alpha_0, \sigma_0] \tag{6.13}$$

where $\alpha_0[\boldsymbol{\varphi}]$, $\sigma_0[\boldsymbol{\varphi}]$ satisfy

$$0 = \frac{1}{N}\frac{\delta\Gamma}{\delta\alpha}\bigg|_{\alpha_0,\sigma_0} = \tfrac{1}{2}\mathrm{i}(\mathrm{i}\sigma_0 + \delta(0)\alpha_0^{-1}) \tag{6.14}$$

and

$$0 = \frac{1}{N}\frac{\delta\Gamma}{\delta\sigma}\bigg|_{\alpha_0,\sigma_0} = \tfrac{1}{2}\mathrm{i}\eta\langle x|(\Box + m_0^2 + \eta\sigma_0)^{-1}|x\rangle - \tfrac{1}{2}\alpha_0 - \frac{\eta}{2N}\boldsymbol{\varphi}^2. \tag{6.15}$$

These equations are no more than

$$\delta A/\delta\sigma = 0 = \delta A/\delta\alpha \tag{6.16}$$

for A of (6.7), on making the substitution (6.11).

We thus need only work with (6.12), in which α, σ are auxiliary fields. The one φ irreducible Green functions are obtained as the functional derivatives of $\Gamma[\boldsymbol{\varphi}, \sigma, \alpha]$, evaluated at the *constant*-field solutions to (6.14) and (6.15), together with the *constant*-field solution to

$$0 = \frac{\delta\Gamma}{\delta\varphi_i} = -(\Box + m_0^2 + \eta\sigma_0)\varphi_i \tag{6.17}$$

i.e. at $\boldsymbol{\varphi} = 0$.

Although the φ fields mix with the σ field in general with

$$\frac{\delta^2\Gamma}{\delta\varphi_i\delta\sigma} = -\eta\varphi_i \tag{6.18}$$

at $\boldsymbol{\varphi} = 0$ they decouple. Thus the two-point function G_2^{ij} is obtained directly as

$$G_2^{ij} = -\frac{\delta^2 W}{\delta h_i \delta h_j}\bigg|_{\boldsymbol{h}=0} = \left(\frac{\delta^2\Gamma}{\delta\varphi_i\delta\varphi_j}\right)^{-1}_{\alpha_0,\sigma_0,\boldsymbol{\varphi}=0}. \tag{6.19}$$

This has the large-N behaviour

$$G_2^{ij}(x)=\delta^{ij}G_2(x) \tag{6.20}$$

where

$$(\Box+m_0^2+\eta\sigma_0)G_2(x)=-\mathrm{i}\delta(x). \tag{6.21}$$

The first property of (6.21) that we notice is that the two-point function G_2 is that of a 'free' theory, insofar that it has no multi-particle continuum, with mass m satisfying

$$m^2=m_0^2+\eta\sigma_0. \tag{6.22}$$

As we had noted earlier, when motivating the lack of normal ordering in the augmented theory, large-N limits get their essential physics from the most singular contributions. However, the 'free' behaviour of G_2 in the large-N limit conventionally arises, not because the theory is truly free, but because it becomes a semiclassical tree theory. In this case the large-N limit of the pseudo-free theory can behave as a free theory. A fuller analysis will be given in a later publication.

It is sufficient for the moment to re-express the constant field solutions to (6.14), (6.15), (6.17) as mass renormalisation for m of (6.22). It follows that

$$\begin{aligned} m^2-m_0^2=\eta\sigma_0&=-\frac{\eta\delta(0)}{\mathrm{i}\alpha_0}=\delta(0)(\langle x|(\Box+m^2)^{-1}|x\rangle)^{-1}\\ &=\delta(0)/\mathrm{i}G_0(0,m^2). \end{aligned} \tag{6.23}$$

This is just equation (4.2), although now *exactly* satisfied to leading order in N^{-1}.

That is, the N^{-1} expansion provides a natural way to separate out the most singular part of the hard-core interactions (or equivalently, the most singular part of the branching equations) due to the scale-invariant measure. We shall not pursue this any further here.

So far we have been so preoccupied with establishing the validity of the formal equation (4.2) (or (6.23)) that we have made no attempt to interpret it. Suppose the pseudo-free theory is in d space–time dimensions. Treated as the $\lambda\to 0$ limit of a $\lambda(\varphi^2)^n$ theory, the failure of the canonical theory forces us to adopt the scale-covariant theory for $d>4n/(2n-2)$, but *a priori* it is defined for all d.

As an intermediate step, we regularise $\delta(0)$ and $G_0(0,m^2)$ by imposing the momentum cut-off $|k|<\Lambda$. This gives

$$\begin{aligned} \frac{\delta(0)_\Lambda}{G_0(0;m^2)_\Lambda}&=\frac{\Lambda^2}{\ln(\Lambda^2/m^2)} && d=2\\ &=\tfrac{1}{3}\Lambda^2+\tfrac{1}{6}\pi\Lambda m+\tfrac{1}{12}\pi^2m^2 && d=3\\ &=\tfrac{1}{2}\Lambda^2-\tfrac{1}{2}m^2\ln(m^2/\Lambda^2) && d=4\\ &=\tfrac{3}{5}\Lambda^2-\tfrac{9}{5}m^2 && d=5\\ &=a_d\Lambda^2+b_dm^2 && (a_d,\ b_d\ \text{finite}) \qquad d>5. \end{aligned} \tag{6.24}$$

We see that for $d>4$ (4.2) has the form (a_d, b_d finite)

$$m^2=(m_0^2+a_d\Lambda^2)(1-b_d)^{-1} \tag{6.25}$$

and the quadratic ultraviolet divergence can be absorbed in m_0^2 to give a finite result on taking $\Lambda \to \infty$. For $d<4$ the presence of logarithmic or linear divergences makes it impossible to express (4.2) in a finite way, and we must assume that the large-N limit of the pseudo-free theory does not exist.

For $d=4$ we still have an expression of the form (6.25) with b_4 logarithmically divergent. We thus need both multiplicative and additive renormalisation of m_0^2 to get a finite result.

This first dimension-specific result is very encouraging since it permits the scale-covariant theory for values of d for which the canonical theory breaks down. However, we expect these results to be modified on including self-interactions. We shall not consider this further here.

7. Conclusions

We summarise our results as follows.

Firstly, our diagrammatic approach has shown that the scale-*invariant* measure has the effect of a non-polynomial effective 'hard-core' interaction.

More subtly, we find that to eschew normal ordering in this effective interaction is equivalent to not performing subtractions in the scale-covariant equations. That is, the diagrams serve as a pictorial representation of the constraint

$$\lim_{y \to x} K_x G_2(x-y) = 0. \tag{7.1}$$

To see this it was necessary to identify the partial mass renormalisation

$$m^2 = m_0^2 + \delta(0)/\mathrm{i}G(0, m^2) \tag{7.2}$$

implied by the most singular part of (6.1) with the mass renormalisation of the diagrammatic expansion. From this point of view the diagrammatic expansion, by the resummation implicit in (7.2), allows the scale of G_2 to remain undetermined, contrary to expectation for a translation-covariant formulation.

Furthermore, we have seen that, assuming factorisation of leading singularities, it is possible to find a consistent solution to the branching equations for the unconnected Green functions that consistently implies (7.1).

What this means is that the branching equations for the scale-covariant theory and the equivalent translation-covariant augmented theory do not obviously mismatch at the coincident point limit of the latter, as might have been expected†.

Finally, we have seen that this separation of most singular parts arises naturally in the large-N limit of the O(N)-invariant pseudo-free theory. For large N, the self-consistent mass renormalisation (7.2) becomes exact. Having thus justified (7.2), we note that it can be re-expressed in terms of finite quantities only for $d \geq 4$ space–time dimensions. It is just for these values that a scale-covariant theory is necessary.

All our results are valid only for the pseudo-free theory. As yet we have very little understanding of the effect of including a $\lambda\varphi^4$ term in the action, to give a generating functional

$$Z'[h] = \int \mathscr{D}[\varphi]\mathscr{D}[\chi] \exp \mathrm{i} \int \mathrm{d}x \left[\tfrac{1}{2}(\partial_\mu \varphi)^2 - \tfrac{1}{2}m_0^2\varphi^2 - \lambda_0\varphi^4 - \tfrac{1}{2}\eta\chi^2\varphi^2 + h\varphi\right]. \tag{7.3}$$

† Since the incompatible operator product expansion and operator-product normal ordering seem, respectively, to be appropriate to the two formalisms (Klauder 1977).

At the very least, we would expect the diagrammatic expansion to correspond to the formal (unsubtracted) equation

$$\lim_{x\to y} K_x G_2(x-y)+4\lambda_0 G_4(xxxx)=0. \tag{7.4}$$

Of course, we expect equation (7.7) (or (7.1)) to be valid only for the case of a scale-*invariant* measure. However, the lesson that we have learned from the arguments above is that one way to proceed is to organise ultraviolet divergences along the lines of the $1/N$ expansion. There is no difficulty, in such an expansion, to work with more general scale-covariant (rather than scale-invariant) measures. This will be discussed in some detail in paper V (Ebbutt and Rivers 1982c).

Appendix. The branching equations

We have already seen that the constraint equations (3.4) are compatible with (indeed motivated) ansatz (5.5). We need to show that equations (3.6) are also satisfied by (5.5), subject to (5.14).

It is convenient to replace (3.6*a*) and (3.6*b*) by the linear combinations (obtained from (3.9))

$$K_x G_{2,0}(x-y)+\eta[G_{2,2}(xy;xx)-G_{2,2}(xx;xy)]=0 \tag{A1}$$

$$\begin{aligned} &\mathrm{i}\sum_r \delta(x-x_r)G_{p,0}(yx_1\ldots\hat{x}_r\ldots x_p)+K_xG_{p+2,0}(xyx_1\ldots x_p)\\ &\qquad+\eta[G_{p+2,2}(xyx_1\ldots x_p;xx)-G_{p+2,2}(xxx_1\ldots x_p;xy)]=0. \end{aligned} \tag{A2}$$

Equation (A2) is the point-split version of the subtracted scale-covariant equations that follow from (3.7) and (A1) the point-split version of the crucial constraint equation (3.10). We retain equations (3.6*c*) for $G_{p,q}$, $q>0$.

We look at those equations in turn, beginning with (A1). With the assumptions (5.1)–(5.4) the equation (A1) becomes

$$K_xG_2(x-y)-\frac{}{G_2(0)}[G_2(x-y)\delta(0)-G_2(0)\delta(x-y)]\approx 0 \tag{A3}$$

or

$$\left[K_x+\frac{\delta(0)}{\mathrm{i}G_2(0)}\right]G_2(x-y)\approx -\mathrm{i}\delta(x-y) \tag{A4}$$

as required. That is, the separability assumption (5.5), for leading singularities, automatically satisfies (3.10).

We move on to the equations (A2). The first of these is

$$\begin{aligned} &\mathrm{i}[\delta(x-x_1)G_2(y-x_2)+\delta(x-x_2)G_2(y-x_1)]+K_xG_4(xyx_1x_2)\\ &\qquad+\eta[G_{4,2}(xyx_1x_2;xx)-G_{4,2}(xxx_1x_2;xy)]\approx 0 \end{aligned} \tag{A5}$$

which becomes, on imposing (5.5),

$$\begin{aligned} &\mathrm{i}[\delta(x-x_1)G_2(y-x_2)+\delta(x-x_2)G_2(y-x_1)]+K_x[G_2(x-y)G_2(x_1-x_2)\\ &\qquad+G_2(x-x_2)G_2(y-x_1)+G_2(x-x_1)G_2(y-x_2)] \end{aligned}$$

$$-\frac{1}{G_2(0)}[\mathrm{i}\delta(0)(G_2(x-y)G_2(x_1-x_2)+G_2(x-x_1)G_2(y-x_1)$$
$$+G_2(x-x_2)G_2(y-x_1))+\mathrm{i}\delta(x-y)G_2(0)G_2(x_1-x_2)]\approx 0. \qquad \text{(A6)}$$

On further rearrangement of terms this gives

$$\left[\left(K_x+\frac{\delta(0)}{\mathrm{i}G_2(0)}\right)G_2(x-y)+\mathrm{i}\delta(x-y)\right]G_2(x_1-x_2)$$
$$+\left[\left(K_x+\frac{\delta(0)}{\mathrm{i}G_2(0)}\right)G_2(x-x_1)+\mathrm{i}\delta(x-x_1)\right]G_2(x_2-y)$$
$$+\left[\left(K_x+\frac{\delta(0)}{\mathrm{i}G_2(0)}\right)G_2(x-x_2)+\mathrm{i}\delta(x-x_2)\right]G_2(y-x_1)\approx 0 \qquad \text{(A7)}$$

which is satisfied automatically by (A4). It takes little effort to see that the remaining equations (A2) are also satisfied.

The first serious test of the justifiable neglect of the right-hand side of (5.12) comes with equations (3.6c), involving more derivatives with respect to j. The first of these is

$$\mathrm{i}\delta(x-x_1)G_{0,q}(y_1\ldots y_q)+K_xG_{2,q}(xx_1;y_1\ldots y_q)+\eta G_{2,q+2}(xx_1;xxy_1\ldots y_q)=0. \qquad \text{(A8)}$$

With the notation $\Delta_q=G_{0,q}$ equation (A8) becomes, on imposing separability,

$$\mathrm{i}\delta(x-x_1)\Delta_q(y_1\ldots y_q)+K_xG_2(xx_1)\Delta_q(y_1\ldots y_q)+\eta G_2(xx_1)\Delta_{q+2}(xxy_1\ldots y_q)\approx 0. \qquad \text{(A9)}$$

Since

$$\Delta_{q+2}(xxy_1\ldots y_q)\approx -\mathrm{i}\delta(0)\Delta_q(y_1\ldots y_q) \qquad \text{(A10)}$$

to leading singularities, (A9) becomes

$$\left[\left(K_x+\frac{\delta(0)}{\mathrm{i}G_2(0)}\right)G_2(xx_1)+\mathrm{i}\delta(x-x_1)\right]\Delta_q(y_1\ldots y_q)\approx 0 \qquad \text{(A11)}$$

which is automatically satisfied.

The second of equations (3.6c) is

$$\mathrm{i}[\delta(x-x_1)G_{2,q}(x_2x_3;y_1\ldots y_q)+\delta(x-x_2)G_{2,q}(x_1x_3;y_1\ldots y_q)$$
$$+\delta(x-x_3)G_{2,q}(x_1x_2;y_1\ldots y_q)]+K_xG_{4,q}(xx_1x_2x_3;y_1\ldots y_q)$$
$$+\eta G_{4,q+2}(xx_1x_2x_3;xxy_1\ldots y_q)\approx 0. \qquad \text{(A12)}$$

On imposing separability as above we have

$$\Big[\mathrm{i}[\delta(x-x_1)G_2(x_2-x_3)+\delta(x-x_2)G_2(x_1-x_3)+\delta(x-x_3)g_2(x_1-x_2)]$$
$$+\left(K_x+\frac{\delta(0)}{\mathrm{i}G_2(0)}\right)[G_2(x-x_1)G_2(x_2-x_3)+G_2(x-x_2)G_2(x_1-x_3)$$
$$+G_2(x-x_2)G_2(x_1-x_2)]\Big]\Delta_q(y_1\ldots y_q)\approx 0 \qquad \text{(A13)}$$

which is again satisfied. With a little work we see that the remaining equations (3.6c) are satisfied, and hence all independent equations are satisfied (to leading singularities).

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